Stochastic PDEs: convergence to the continuum?

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A stochastic partial differential equation (stochastic PDE) describes dynamics with noise in continuous space and time. On discretising space to generate a numerical solution, one obtains a set of coupled stochastic ordinary differential equations.

In the finite differences method, independent mean-zero gaussian random variables are added at each timestep and at each grid point, with variance proportional to the length of the timestep Δt and to $\Delta x^{-m/2}$, where Δx is the distance between grid points and m the number of space dimensions. In one space dimension, the continuum solution takes values in a space of continuous functions. That is, in a fixed realisation at a given time, one obtains a configuration that is a continuous function of space. In more than one space dimension, the configurations are not necessarily continuous functions but only distributions. Such stochastic PDEs can nevertheless be solved on a finite grid of points in space; however the mean squared value at a grid point does not approach a finite limit as the grid spacing is decreased.

The linear stochastic PDE known as the infinite-dimensional Ornstein- Uhlenbeck process is commonly used as spacetime coloured noise: in the same way as the standard Ornstein-Uhlenbeck stochastic differential equation is used as an auxiliary equation to add coloured noise to an ordinary differential equation, the linear stochastic PDE can be used to generate noise with non-delta function correlations in both space and time. The initial value problem can be solved analytically in finite differences and in the continuum limit. A Fourier transform in space separates the stochastic PDE into a series of uncoupled SDEs for the Fourier coefficients.

In one space dimension, where the continuum solution is a stochastic process whose values are continuous functions in space, the transfer integral allows exact calculation of equilibrium properties, including the corrections due to finite grid spacing. The method applies to arbitrarily nonlinear PDEs, provided they have a stationary density. Consider stochastic PDEs of the following form:

$$\mathrm{d}\boldsymbol{\phi}_t(x) = f(\boldsymbol{\phi}_t(x))\mathrm{d}t + \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}\boldsymbol{\phi}_t(x)\mathrm{d}t + (\frac{2}{\beta})^{\frac{1}{2}}\,\mathrm{d}\mathbf{W}_t(x), \tag{1}$$

where $\boldsymbol{\phi}_t(x)$ is scalar valued and $f(x)$ is a nonlinear function. A general time-dependent solution cannot be written down, but a steady state solution can be shown to exist under fairly general conditions and can be analysed using the transfer integral [2]. The steady state density of the field at a point is given as follows. Let ϵ_n and $\psi_n(u)$ be the eigenvalues and corresponding normalised eigenfunctions of the equation

$$\left(-\frac{1}{2\beta^2}\frac{\partial^2}{\partial u^2} + V(u)\right)\psi_n(u) = \epsilon_n\psi_n(u), \tag{2}$$

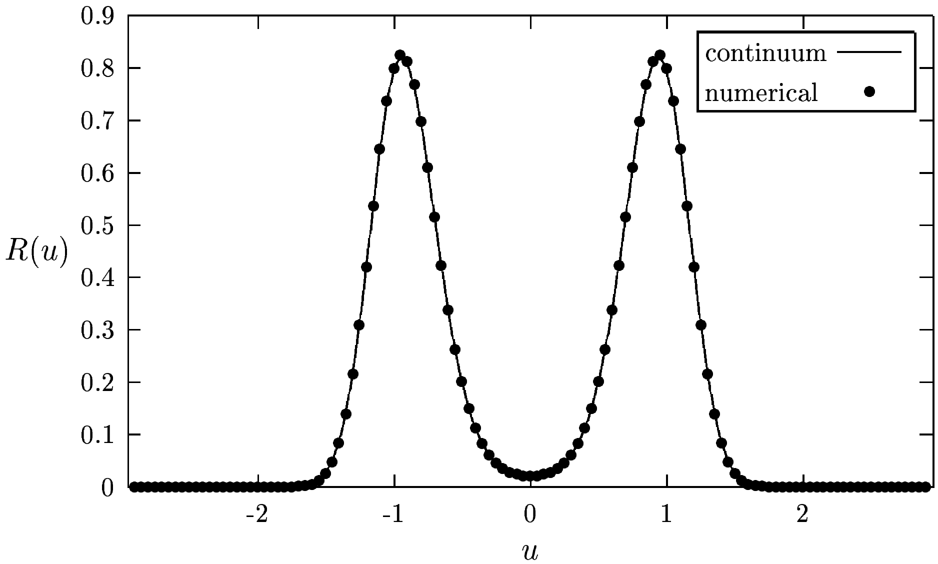

Figure 1: *Steady state density* for $\beta = 7$. The solid line is $\psi_0(u)^2$, predicted from the first eigenfunction of (2). The dots are a histogram obtained from a numerical solution of the stochastic PDE run with grid spacing $\Delta x = 0.2$.

where

$$\frac{\mathrm{d}}{\mathrm{d}u} V(u) = -f(u) \tag{3}$$

and $n = 0$ corresponds to the eigenfunction with the smallest eigenvalue. Let

$$R(u) = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}u} \mathcal{P}[\boldsymbol{\phi}_t(x) < u]. \tag{4}$$

Then $R(u) = \psi_0(u)^2$.

Equation (2) can be modified to include powers of Δx^2. The lowest-order correction to the continuum is obtained by replacing the on-site potential $V(u)$ by [2]

$$U(u, \Delta x) = V(u) - \frac{\Delta x^2}{24} V'(u)^2. \tag{5}$$

References

[1] Salman Habib and Grant Lythe, Phys. Rev. Lett. **84** 1070 (2000)

[2] Luis M. A. Bettencourt, Salman Habib and Grant Lythe, Physical Review D **60** 105039 (1999)