NOISE ASSISTED HIGH-GAIN STABILIZATION: ALMOST SURELY VERSUS IN SECOND MEAN

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Abstract

The dependence of dynamical properties of systems on parameters is a central question for many problems in control theory and in dynamical systems. For example, consider a linear single-input single-output control system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= C^T x \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, A is a real $n \times n$-matrix, and $B, C \in \mathbf{R}^n$. A basic control design technique is to study the root locus for (1), i. e., the eigenvalues of $A - kBC$, as k varies. The root locus technique is used, for example, to show that if $C^T B > 0$, and if (A, B, C) is minimum phase, then high gain feedback control $u = -ky$ is stabilizing in the sense that all eigenvalues of $A - kBC$ are in the left half-plane for all k sufficiently large.

The dependence of dynamical properties of linear *stochastic* differential equations on parameters has been investigated intensely during the last decades. Here we investigate a particular class of problems in this field, namely the dependence of dynamical properties of certain LSDEs on proportional feedback. More precisely, we consider a noisy version

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of (1) with proportional output feedback $u = -ky$, which gives the system

$$dx = (A - kBC)x\,dt + \sum_{j=1}^{m} A_j x \circ dW_j(t)\,. \tag{2}$$

We are interested in the dependence of exponential growth rates for (2) on k, and, in particular, on asymptotic formulas, valid for large k. Whilst for deterministic systems this can be derived by studying the dependence of the eigenvalues on parameters, for LSDEs the situation is more complicated. Indeed, there are several competing notions of growth rates (for example, in p^{th} mean or almost surely).

We first recall some basic facts about LSDEs and their exponential growth rates, specifically the leading Lyapunov exponent λ and p^{th} moment growth rates $g(p)$ for p real. It is well known that $g(p)/p \geq \lambda$ for $p \geq 0$. In particular, $g(p) < 0$ for some $p > 0$ implies $\lambda < 0$, which means that p^{th} mean stability for some $p > 0$ implies almost sure stability.

Next we restrict attention to the simplest possible non-trivial case, which is the case of a 2×2-system. First we consider almost sure growth rates, and we characterize under which conditions (2) can be stabilized almost surely by high gain feedback. This means that the leading Lyapunov exponent becomes negative for sufficiently high gain k. To calculate the leading Lyapunov exponent we use the Furstenberg-Khasminskii formula, which yields a closed, albeit complicated, formula. Next we consider 2^{nd} mean growth rates and characterize under which conditions the system can be stabilized in second mean by high gain feedback. Here we use the classical technique of considering the norm induced by a positive definite matrix P (thus defining a Lyapunov function), applying the Itô formula to obtain estimates for the exponential growth of second moments, and then choosing the matrix P in an appropriate way.

As a result we obtain the following. If the noise enters the system in a purely skew-symmetric way, then for high gain the exponential growth rate of the second moment approaches twice the Lyapunov exponent, the almost sure exponential growth rate. If, however, the noise is entering not just skew-symmetrically, but has also diagonal terms, then the growth rates have different limiting behaviour as $k \to \infty$. In particular, we find cases where an increase of the noise causes almost sure high-gain stabilizability, and at the same time the same increase of the noise destabilizes with respect to second mean in the high-gain region.