RESPONSE SPECTRAL DENSITIES OF SYSTEMS UNDER BOTH ADDITIVE AND MULTIPLICATIVE EXCITATIONS

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SUMMARY

Consider a stochastically excited system governed by the following equations

$$\ddot{q}_j + h_j(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{i=1}^{n} [a_{ji}\dot{q}_i\eta_i(t) + b_{ji}q_i\gamma_i(t)] + \xi_j(t) \qquad (j = 1, 2, ..., n) \tag{1}$$

where $h_j(\boldsymbol{q}, \dot{\boldsymbol{q}})$ include both damping and stiffness forces, $\eta_i(t), \gamma_i(t)$ and $\xi_j(t)$ are Gaussian white noises, and where multiplicative excitations appear in the linear terms. The main objective is to determine the spectral density functions of the system response.

The case of linear damping and stiffness forces is considered first, namely, $h_j(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{i=1}^{n} (\alpha_{ji}\dot{q}_i + \beta_{ji}q_i)$. Such a system is quasi-linear since multiplicative excitations are present. Letting $q_j = X_{2j-1}$ and $\dot{q}_j = X_{2j}$, a set of Ito stochastic differential equations is derived from (1) as follows

$$\begin{aligned} dX_{2j-1} &= X_{2j}\, dt \\ dX_{2j} &= \sum_{i=1}^{2n} C_{ji} X_i\, dt + \sigma_j(\boldsymbol{X}) dB_j(t) \end{aligned} \qquad (j = 1, 2, ..., n) \tag{2}$$

where $B_j(t)$ are independent unit Wiener processes, and the Wong-Zakai correction terms have been incorporated. Denoting $m_{ij} = \mathrm{E}[X_i X_j]$, and using the Ito stochastic differential rule, a set of ordinary differential equations for the second-order statistical moments m_{ij} can be obtained. It is noticed that these equations are closed in the second order, and they can be solved exactly to obtain either transient or stationary second-order moments. If higher-order moments are needed, they can be obtained similarly.

Multiplying $X_k(t_0), t_0 = t - \tau,$ on both sides of (2) and taking assemble average, we obtain a set of equations for the correlation functions $R_{ij}(\tau) = \mathrm{E}[X_i(t) X_j(t_0)]$ as follows

$$\begin{aligned} \frac{dR_{2j-1,k}(\tau)}{d\tau} &= R_{2j,k}(\tau) \\ \frac{dR_{2j,k}(\tau)}{d\tau} &= \sum_{i=1}^{2n} C_{ji} R_{ik}(\tau) \end{aligned} \qquad (j = 1,2,...,n;\ k = 1,2,...,2n) \tag{3}$$

Equations (3) can be solved with initial conditions $R_{ij}(0) = m_{ij}$, which are assumed to have been obtained from the equations for the second-order moments. The spectral density functions can then be obtained as the Fourier transform of the correlation functions. However, if only the spectral densities are required, a direct procedure is given below.

Define the following integral transformation

$$\overline{\Phi}_{ij}(\omega) = \Im[R_{ij}(\tau)] = \frac{1}{\pi}\int_0^\infty R_{ij}(\tau)e^{-i\omega\tau}d\tau \tag{4}$$

It can be shown that

$$\Im\left[\frac{dR_{ij}(\tau)}{d\tau}\right] = i\omega\overline{\Phi}_{ij}(\omega) - \frac{1}{\pi}m_{ij} \tag{5}$$

Using (4) and (5), (3) can be transformed to

$$\begin{aligned} i\omega\overline{\Phi}_{2j-1,k}(\omega) - \frac{1}{\pi}m_{2j-1,k} &= \overline{\Phi}_{2j,k}(\omega) \\ i\omega\overline{\Phi}_{2j,k}(\omega) - \frac{1}{\pi}m_{2j,k} &= \sum_{i=1}^{2n} C_{ji}\overline{\Phi}_{ik}(\omega) \end{aligned} \qquad (j = 1,2,\ldots,n;\ k = 1,2,\ldots,2n) \tag{6}$$

The set of $\Phi_{ij}(\omega)$ can be solved from (6). The spectral density functions are then obtained from

$$\Phi_{ii}(\omega) = \mathrm{Re}[\overline{\Phi}_{ii}(\omega)], \quad \Phi_{ij}(\omega) = \frac{1}{2}[\overline{\Phi}_{ij}(\omega) + \overline{\Phi}^*_{ji}(\omega)] \tag{7}$$

It is noticed that (6) is a set of linear algebraic equations and analytical solutions can be obtained for low dimensional systems. For high dimensional cases, analytical solutions are tedious although numerical solutions can be carried out quite simply. The above procedure was used to treat a nonlinear oscillator under an additive excitation (Cai and Lin, 1997).

For cases of nonlinear damping and stiffness forces, statistical linearization procedure can be applied to replace these nonlinear forces by linear ones, with all the excitation terms unchanged. Since the exact moments are available for the substituting quasi-linear system, they can be used in the linearization without assuming a Gaussian distribution as in the conventional linearization. The preservation of the parametric excitations is an important feature of the approximation procedure.

Numerical calculations are carried out for a system with both nonlinear damping and stiffness forces. It is shown that the results obtained from the present approximation procedure are quite accurate when compared with those obtained from Monte Carlo simulations.

Cai, G. Q. and Lin, Y. K., 1997, "Response Spectral Densities of Strongly Nonlinear Systems under Random Excitation", *Probabilistic Engineering Mechanics*, 12(1),41-47.