Stochastic Averaging: some methods and proofs

P. Bernard

Laboratoire de Mathématiques Appliquées
Université Blaise Pascal
63 177 Aubière Cedex, France

This work is dedicated to Professor J.B.Roberts.

When studying the dynamics of a structure submitted to some actions of stochastic type, if the excitation, related to the characteristics of the structure, is not of too high level, the dynamical geometric properties of the structure will be determinant in the response analysis.

In such cases, the dynamics of a structure with non-linearities can be modelled as the dynamics of an Hamiltonian system with a small dissipative perturbation. If the random exterior actions are not of too high level, then we are concerned with the study of an Hamiltonian system perturbed by dissipative and stochastic terms.

Our study concerns highly nonlinear systems, as far as there is some information available about the dynamics of the unperturbed one. In this contribution, we only deal with dimension one.

Rescaling time, then introducing a change of variables whose definition is a crucial part of the method, a slow and a fast process appear. Then the idea of averaging can be used. This idea goes back to J.L.Lagrange, in his "Mécanique Analytique", published in 1788. It is in fact the well known "variation des constantes" method. It gives an exact method of resolution of first order linear nonhomogeneous ordinary differential equations, but it was in fact introduced as a perturbation method. After there intoduction in Astronomy (Clairaut, Lagrange, Laplace), these methods were extanded to nonlinear problems in dynamical systems (Jacobi, Poincaré, van der Pol). Stratonovitch gave the first stochastic version at the level of rigor of physics, and Hasminskii the first proof of a Law of Large Numbers, a Central Limit Theorem and a Diffusion Approximation result.

Results by Hasminskii [5],[6] were given under very drastic conditions. In particular, all coefficients were expectd to be bounded. Moreover, there was a mixing condition rather difficult to check.

The results presented in this paper are obtained through very weak conditions, which can be verified in practical applications. Noise is White Noise, but the results are available in the case of parametric stochastic excitation. As the method rests on the martingale formulation of the problem, the assumption is that the martingale problem is well posed. In the case of diffusion processes, this is a non-explosion of solutions condition. One cannot hope weaker assumptions.

By the modified test functions approach, as introduced by Papanicolaou, Stroock, Varadhan, Kurtz, Kushner, diffusion approximation results are proved concerning the slow process (which usually is not a diffusion process). These results extend Hasminskii's results.

Using this limit averaged diffusion process as an approximation of the slow process, and some results concerning the asymptotic behavior (with respect to level) of level crossings by a one dimensional diffusion process, approximations of the probability distribution of the maximum of the absolute value of the displacement of a strogly nonlinear oscillator are obtained. These formulas, compared with other formulas in the litterature, proved to be much better (numerical tests are due to S.Espinouze [3]).

References

[1] N.Bogoliubov & Y.Mitropolsky. Asymptotic Methods in the Theory of Nonlinear Oscillations. *Gordon & Breach (1961).*

[2] A.N.Borodin. A limit theorem for solutions of differential equations with random right hand sides. *Theory of Probability and its Applications, Vol XXII, No 3, pp.482-497, (1977).*

[3] S.Espinouze. *PHD Thesis. Blaise Pascal University, (2001).*

[4] M.I. Freidlin & A.D. Wentzell. Random Perturbations of Dynamical Systems. *Springer Verlag (1984)*

[5] R.Z. Khas'minskii. On stochastic processes defined by differential equations with a small parameter. *Theory of Probability and its Applications, Vol XI, No 2, pp.211-228 (1966).*

[6] R.Z. Khas'minskii. A limit theorem for the solutions of differential equations with random right hand side. *Theory of Probability and its Applications, Vol XI, No 3, pp.390-406 (1966).*

[7] Z.L.Huang, W.Q.Zhu, Y.Suzuki. Stochastic averaging of strongly non-linear oscillators under combined harmonic and white-noise excitations. *Jal of Sound and Vibrations, 238 (2), pp. 233-256 (2000).*

[8] J.L.Lagrange. Mécanique Analytique. *A.Blanchard, Paris (1788).*

[9] G.C. Papanicolaou, D. Stroock & S.R.S Varadhan. Martingale Approach to Some Limit Theorems. *Duke Turbulence Conference (1976).*

[10] A.Rachad. *PHD Thesis. Blaise Pascal University,*(1999).

[11] J.R. Red Horse, P.D.Spanos: A generalization to stochastic averaging in random vibrations. *International Journal of Nonlinear Mechanics*, 27, 1, 85-101, (1992).

[12] J.B.Roberts. First passage time for oscillators with nonlinear restoring force. *Jal of Sound and Vibration*, 56, 1, 71-86, (1978).

[13] J.B.Roberts & P.D.Spanos, Stochastic Averaging: an approximate method of solving random vibration problems. *Int. J. Non-Linear Mechanics*, 21, 2, 111-134, (1986).

[14] J.A.Sanders & F.Vershulst, Averaging Methods in Nonlinear Dynamical Systems. *Springer Verlag, (1965)*

[15] R.L Stratonovitch, Topics in the theory of random noise, *Vols 1 and 2, Gordon & Breach, New York* (1963).